

Scalar products of symmetric functions and matrix integrals¹

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Abstract

We present relations between Hirota-type bilinear operators, scalar products on spaces of symmetric functions and integrals defining matrix model partition functions. Using the fermionic Fock space representation, a proof of the expansion of an associated class of KP and 2-Toda tau functions $\tau_{r,n}$ in a series of Schur functions generalizing the hypergeometric series is given and related to the scalar product formulae. It is shown how special cases of such τ -functions may be identified as formal series expansions of partition functions. A closed form expansion of $\log \tau_{r,n}$ in terms of Schur functions is derived.

1 Partitions, Schur functions and scalar products

1.1 Partitions

We recall here some standard definitions and notations concerning symmetric functions and partitions (see [10] for further details). Symmetric polynomials of many variables are parameterized by partitions. A *partition* is any (finite or infinite) sequence of non-negative integers in decreasing order:

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \dots), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq \dots \quad (1-1)$$

The nonzero λ_i 's in (1-1) are called the *parts* of λ ; the number of parts is the *length* of λ , denoted $l(\lambda)$, and the sum of the parts is the *weight* of λ , denoted by $|\lambda|$. If $n = |\lambda|$ we say that λ is a *partition of n* . The partition of zero is denoted (0) .

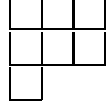
The *diagram* of a partition (or *Young diagram*) may be defined as the set of points (or nodes) $\{(i, j) \in \mathbf{Z}^2\}$ such that $1 \leq j \leq \lambda_i$. Thus a Young diagram may be viewed as a subset of entries in a matrix with $l(\lambda)$ rows and λ_1 columns. We shall denote the diagram of λ by the same symbol.

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For example

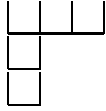


is the diagram of $(3, 3, 1)$. The weight of this partition is 7 and the length is 3.

Another notation for partitions is due to Frobenius. Suppose that the main diagonal of the diagram of λ consists of r nodes (i, i) ($1 \leq i \leq r$). Let $\alpha_i = \lambda_i - i$ be the number of nodes in the i th row of λ to the right of (i, i) , for $1 \leq i \leq r$, and let $\beta_i = \lambda'_i - i$ be the number of nodes in the i th column of λ below (i, i) , for $1 \leq i \leq r$. We have $\alpha_1 > \alpha_2 > \dots > \alpha_r \geq 0$ and $\beta_1 > \beta_2 > \dots > \beta_r \geq 0$. In Frobenius' notation the partition λ is denoted

$$\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r) = (\alpha | \beta) . \quad (1-2)$$

This corresponds to a hook decomposition of the diagram of λ , where the biggest hook is $(\alpha_1 | \beta_1)$, the next is $(\alpha_2 | \beta_2)$, and so on, down to the smallest one, which is $(\alpha_r | \beta_r)$. The corners of the hooks are situated on the main diagonal of the diagram. For instance the partition $(3, 3, 1)$ consists of two hooks $(2, 2)$ and $(1, 0)$,



and



so, in Frobenius notation this is $(2, 1 | 2, 0)$.

1.2 Scalar products of Schur functions

In applications to KP theory it is useful to view the Schur function s_λ as a basis in the space of weighted homogeneous polynomials in an infinite sequence of parameters $\gamma = (\gamma_1, \gamma_2, \dots)$ (with $\text{weight}(\gamma_j) = j$) defined by

$$s_\lambda(\gamma) = \det(h_{\lambda_i - i + j}(\gamma))_{1 \leq i, j \leq r}, \quad (1-3)$$

where $\{h_n(\gamma)\}_{n \in \mathbf{Z}}$ are the elementary Schur functions (or complete symmetric functions) defined by the Taylor expansion

$$e^{\xi(\gamma, z)} := \exp\left(\sum_{k=1}^{\infty} \gamma_k z^k\right) = \sum_{n=0}^{\infty} z^n h_n(\gamma) , \quad (1-4)$$

and $h_n(\gamma) := 0$ if $n < 0$. If the parameters $\gamma = (\gamma_1, \gamma_2, \dots)$ are determined in terms of a finite number of variables (x_1, \dots, x_N) by

$$\gamma_j = \frac{1}{j} \sum_{a=1}^N x_a^j , \quad (1-5)$$

i.e., if the Schur functions are interpreted as irreducible characters for $GL(N)$, we use the notation $s_\lambda([\mathbf{x}])$, where

$$[\mathbf{x}] := \left(\sum_{a=1}^N x_a, \sum_{a=1}^N x_a^2, \dots \right) . \quad (1-6)$$

The Cauchy-Littlewood identity [10] provides a generating function formula for the full set of Schur functions, viewed as functions of two finite sets of variables (x_1, \dots, x_N) , (y_1, \dots, y_M) :

$$\prod_{a=1}^N \prod_{b=1}^M (1 - x_a y_b)^{-1} = \sum_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) , \quad (1-7)$$

where

$$t_j = \frac{1}{j} \sum_{a=1}^N x_a^j, \quad t_j^* = \frac{1}{j} \sum_{a=1}^M y_a^j , \quad (1-8)$$

and $\mathbf{t} := (t_1, t_2, \dots)$ and $\mathbf{t}^* := (t_1^*, t_2^*, \dots)$.

A scalar product [10] is defined on the space of weighted homogeneous polynomials in an infinite sequence of variables, such that the Schur functions are orthonormal

$$\langle s_{\lambda}, s_{\mu} \rangle = \delta_{\mu, \lambda} . \quad (1-9)$$

In the context of the KP and Toda lattice hierarchies the variables $\{x_i\}$ in (1-5) (possibly with $N = \infty$) are called the Hirota-Miwa variables while $\gamma = (\gamma_1, \gamma_2, \dots)$ play the role of KP flow parameters.

2 A deformation of the standard scalar product and series of hypergeometric type

2.1 Scalar product $\langle, \rangle_{r,n}$

Given a function $r(n)$ defined on the integers $n \in \mathbf{Z}$ and a partition λ of weight $|\lambda|$ and length $l(\lambda)$, let us define

$$r_{\lambda}(n) := \prod_{i,j \in \lambda} r(n+j-i) = \prod_{i=1}^{l(\lambda)} \prod_{j=1}^{\lambda_i} r(n+j-i). \quad (2-1)$$

Thus $r_{\lambda}(n)$ is a product of the values of the function r translated by the *content* $(j-i)$ of the node, taken over all nodes of the Young diagram of the partition λ . For instance, for the partition $(3, 3, 1)$,

$$r_{(3,3,1)}(n) = r(n+2)(r(n+1))^2(r(n))^2r(n-1)r(n-2) . \quad (2-2)$$

For the zero partition one puts $r_0 := 1$.

Given a function r and an integer n which is greater than or equal to the largest zero of r , we may define an associated bilinear form $\langle, \rangle_{r,n}$ as follows:

$$\langle s_{\lambda}, s_{\mu} \rangle_{r,n} = r_{\lambda}(n) \delta_{\lambda\mu} . \quad (2-3)$$

Let $\{n_i \in \mathbf{Z}\}$ be the zeros of r and

$$k := \min |(n - n_i)|. \quad (2-4)$$

The product (2-3) is nondegenerate on the space Λ_k of symmetric polynomials in k variables $\mathbf{x}^k := (x_1, \dots, x_k)$. This follows from the fact that, from the definition (2-1), the quantity $r_{\lambda}(n)$ never vanishes for partitions λ with length $l(\lambda) \leq k$. In this case the Schur functions of k variables $\{s_{\lambda}([\mathbf{x}^k]), l(\lambda) \leq k\}$ form a basis for Λ_k . If, on the contrary, $n - n_i < 0$ for all zeros of r , then the factor $r_{\lambda}(n)$ never vanishes for the partitions $\{\lambda : l(\lambda') \leq k\}$, where λ' is the conjugate partition, and $\{s_{\lambda}(-[\mathbf{x}^k]), l(\lambda') \leq k\}$ form a basis on Λ_k . If r is a nonvanishing function then the scalar product is non-degenerate on Λ_{∞} .

2.2 The main example.

The case $r(n) = n$ plays a special role in applications to two-matrix models. We denote by \langle, \rangle'_n the scalar product for this case

$$\langle s_\lambda, s_\mu \rangle'_n = (n)_\lambda \delta_{\lambda\mu}, \quad (n)_\lambda = \prod_{i=1}^{l(\lambda)} \prod_{j=1}^{\lambda_i} (n - i + j). \quad (2-5)$$

(Recall that the restriction of this to any space of symmetric polynomials in $\leq n$ variables is non-degenerate.) For a pair (f, g) of symmetric functions of n variables $\mathbf{x} = (x_1, \dots, x_n)$, we have the following simple realization of this scalar product.

$$\langle f, g \rangle'_n = \frac{1}{c_n} \Delta(\partial) f(\partial) \cdot \Delta(\mathbf{x}) g(\mathbf{x})|_{\mathbf{x}=0}, \quad c_n = \prod_{k=1}^n k! \quad (2-6)$$

where $\Delta(\mathbf{x}) := \prod_{1 \leq i < j \leq n} (x_i - x_j)$ is the Vandermonde determinant,

$$\Delta(\partial) := \prod_{1 \leq i < j \leq n} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right), \quad (2-7)$$

and $f(\partial)$ is the operator obtained by replacing $\{x_i\}$ by $\{\frac{\partial}{\partial x_i}\}$. The proof follows from the Jacobi-Trudi formula for the Schur function [10]

$$s_\lambda(\mathbf{x}) = \det (x_i^{\lambda_j + n - j})_{i,j=1,\dots,n} / (\Delta(\mathbf{x})), \quad (2-8)$$

and the formula

$$\frac{1}{c_n} \det (\partial_{x_i}^{\lambda_j + n - j})_{i,j=1,\dots,n} \cdot \det (x_i^{\mu_j + n - j})_{i,j=1,\dots,n} |_{\mathbf{x}=0} = (n)_\lambda \delta_{\lambda\mu} = \langle s_\lambda, s_\mu \rangle'_{r,n} \quad (2-9)$$

(It is interesting to compare the scalar product (2-6) with the one considered in [5].)

We present two different realizations of the formula (2-6) via multiple integrals.

(A)

$$\langle f, g \rangle'_n = \frac{1}{\pi^n c_n} \int_C \dots \int_C f(\mathbf{z}) g(\bar{\mathbf{z}}) e^{-|z_1|^2 - \dots - |z_n|^2} |\Delta(\mathbf{z})|^2 d^2 z_1 \dots d^2 z_n, \quad (2-10)$$

where the integration is over n copies $\{z_i\}_{i=1\dots n}$ of the complex plane.

The proof of (2-10) follows from the following relation, valid for any pair of polynomial functions (a, b) of one variable

$$a\left(\frac{\partial}{\partial z}\right) \cdot b(z)|_{z=0} = \frac{1}{\pi} \int_C a(z) b(\bar{z}) e^{-|z|^2} dz d\bar{z}. \quad (2-11)$$

(B)

$$\langle f, g \rangle'_n = \frac{1}{(2\pi i)^n c_n} \int_{\mathbb{R}} \int_{\Im} \dots \int_{\mathbb{R}} \int_{\Im} f(\mathbf{x}) g(\mathbf{y}) e^{-(x_1 y_1 + \dots + x_n y_n)} \Delta(\mathbf{x}) \Delta(\mathbf{y}) dx_1 dy_1 \dots dx_n dy_n, \quad (2-12)$$

where the integration is over n copies $\{x_i\}_{i=1\dots n}$ of the real line and n copies $\{y_i\}_{i=1\dots n}$ of the imaginary line. If f and g are polynomials, these integrals must be evaluated in the sense of distributions, with a suitable regularization procedure. Alternatively, we may interpret (2-12) as applied to a pair of functions f, g that decrease rapidly enough at ∞ to make the integral converge.

The proof of (2-12) follows from the formula

$$a\left(\frac{\partial}{\partial x}\right) \cdot b(x)|_{x=0} = \frac{1}{2\pi i} \int_{\Re} \int_{\Im} a(x)b(y)e^{-xy} dx dy . \quad (2-13)$$

Below we shall mainly be concerned with the following series

$$\frac{1}{c_n} e^{\sum_{m=0}^{\infty} \sum_{i=1}^n t_m \partial_{x_i}^m} \Delta(\partial) \cdot e^{\sum_{m=0}^{\infty} \sum_{i=1}^n t_m^* x_i^m} \Delta(x)|_{\mathbf{x}=0} = \sum_{\lambda} (n)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) . \quad (2-14)$$

This equality follows from (2-5), (2-6) and the generalized Cauchy-Littlewood identity (see (4-12) below). The r.h.s. is generally divergent, and must be interpreted as a formal graded polynomial series. It may be identified with the hypergeometric series ${}_2\mathcal{F}_0(n, n; \mathbf{t}, \mathbf{t}^*)$, see [14].

3 Fermionic operators, vacuum expectations, tau functions

3.1 Free fermions

In this section we use the formalism of fermionic Fock space as in [2]. The algebra of *free fermions* is the infinite dimensional Clifford algebra \mathbf{A} over \mathbf{C} with generators $\psi_n, \psi_n^* (n \in \mathbf{Z})$ satisfying the anti-commutation relations:

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0; \quad [\psi_m, \psi_n^*]_+ = \delta_{mn} , \quad m, n \in \mathbf{Z} . \quad (3-1)$$

Any element of $W = (\oplus_{m \in \mathbf{Z}} \mathbf{C} \psi_m) \oplus (\oplus_{m \in \mathbf{Z}} \mathbf{C} \psi_m^*)$ will be referred to as a *free fermion*. The Clifford algebra has a standard representation (*Fock representation*) as follows. Put $W_{an} = (\oplus_{m < 0} \mathbf{C} \psi_m) \oplus (\oplus_{m \geq 0} \mathbf{C} \psi_m^*)$, and $W_{cr} = (\oplus_{m \geq 0} \mathbf{C} \psi_m) \oplus (\oplus_{m < 0} \mathbf{C} \psi_m^*)$, and consider the left (resp. right) \mathbf{A} -module $F = \mathbf{A}/\mathbf{A}W_{an}$ (resp $F^* = W_{cr}\mathbf{A}/\mathbf{A}$). These are cyclic \mathbf{A} -modules generated by the vectors $|0\rangle = 1 \bmod \mathbf{A}W_{an}$ (resp. by $\langle 0| = 1 \bmod W_{cr}\mathbf{A}$), with the properties

$$\psi_m |0\rangle = 0 \quad (m < 0), \quad \psi_m^* |0\rangle = 0 \quad (m \geq 0), \quad (3-2)$$

$$\langle 0| \psi_m = 0 \quad (m \geq 0), \quad \langle 0| \psi_m^* = 0 \quad (m < 0). \quad (3-3)$$

Vectors $\langle 0|$ and $|0\rangle$ are referred to as left and right vacuum vectors. Fermions $w \in W_{an}$ annihilate the left vacuum vector, while fermions $w \in W_{cr}$ annihilate the right vacuum vector.

The Fock spaces F and F^* are mutually dual, with the pairing defined through the linear form $\langle 0||0\rangle$ on \mathbf{A} called the *vacuum expectation value*. This is given by

$$\langle 0|1|0\rangle = 1, \quad \langle 0|\psi_m \psi_m^* |0\rangle = 1 \quad m < 0, \quad \langle 0|\psi_m^* \psi_m |0\rangle = 1 \quad m \geq 0, \quad (3-4)$$

$$\langle 0|\psi_m \psi_n |0\rangle = \langle 0|\psi_m^* \psi_n^* |0\rangle = 0, \quad \langle 0|\psi_m \psi_n^* |0\rangle = 0 \quad m \neq n, \quad (3-5)$$

and by **the Wick rule**, which is

$$\langle 0|w_1 \cdots w_{2n+1}|0\rangle = 0, \quad \langle 0|w_1 \cdots w_{2n}|0\rangle = \sum_{\sigma} \text{sgn} \sigma \langle 0|w_{\sigma(1)} w_{\sigma(2)} |0\rangle \cdots \langle 0|w_{\sigma(2n-1)} w_{\sigma(2n)} |0\rangle, \quad (3-6)$$

where $w_k \in W$, and σ runs over permutations such that $\sigma(1) < \sigma(2) \cdots \sigma(2n-1) < \sigma(2n)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$.

3.2 The Lie algebra $\widehat{gl}(\infty)$

Consider infinite matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ satisfying the condition that there exists an N such that $a_{ij} = 0$ for $|i - j| > N$. Such matrices are called generalized Jacobi (or “finite band”) matrices, and form a Lie algebra under the usual matrix commutator bracket.

Let $:$ \dots $:$ denote the normal ordering operator, defined such that

$$:\psi_i \psi_j^* := \psi_i \psi_j^* - \langle 0 | \psi_i \psi_j^* | 0 \rangle. \quad (3-7)$$

The space of linear combinations of quadratic elements of the form

$$\sum_{i,j \in \mathbb{Z}} a_{ij} : \psi_i \psi_j^* : , \quad (3-8)$$

together with the identity element 1, span the infinite dimensional Lie algebra $\widehat{gl}(\infty)$:

$$[\sum a_{ij} : \psi_i \psi_j^* :, \sum b_{ij} : \psi_i \psi_j^* :] = \sum c_{ij} : \psi_i \psi_j^* : + c_0 . \quad (3-9)$$

where

$$c_{ij} = \sum_k a_{ik} b_{kj} - \sum_k b_{ik} a_{kj}, \quad (3-10)$$

The last term

$$c_0 = \sum_{i < 0, j \geq 0} a_{ij} b_{ji} - \sum_{i \geq 0, j < 0} a_{ij} b_{ji}. \quad (3-11)$$

is central, so the Lie algebra $\widehat{gl}(\infty)$ is a central extension of the algebra of generalized Jacobian matrices.

3.3 Bilinear identity

Let g be the exponential of an operator in $\widehat{gl}(\infty)$:

$$g = \exp \sum_{i,j \in \mathbb{Z}} a_{ij} : \psi_i \psi_j^* : . \quad (3-12)$$

i.e., an element of the corresponding group. Using (3-9) it is possible to derive the following relation

$$g \psi_n = \sum_m \psi_m A_{mn} g , \quad \psi_n^* g = g \sum_m A_{nm} \psi_m^* , \quad (3-13)$$

where the coefficients A_{nm} are determined by a_{nm} . In turn (3-13) implies [2]

$$[\sum_{n \in \mathbb{Z}} \psi_n \otimes \psi_n^*, g \otimes g] = 0 . \quad (3-14)$$

This last relation is very important for applications to integrable systems, and is equivalent to the Hirota bilinear equations.

3.4 Double KP flow parameters. The KP and TL tau functions.

We introduce the following operators.

$$H_n = \sum_{k=-\infty}^{\infty} \psi_k \psi_{k+n}^*, \quad n \neq 0, \quad H(\mathbf{t}) = \sum_{n=1}^{+\infty} t_n H_n, \quad H^*(\mathbf{t}^*) = \sum_{n=1}^{+\infty} t_n^* H_{-n}. \quad (3-15)$$

Here, $H_n \in \widehat{gl}(\infty)$, and $H(\mathbf{t}), H^*(\mathbf{t}^*)$ also belong to $\widehat{gl}(\infty)$ if we restrict the number of non-vanishing parameters $\{t_m, t_m^*\}$ to be finite. For H_n we have the Heisenberg algebra commutation relations:

$$[H_n, H_m] = n\delta_{m+n,0}. \quad (3-16)$$

Note also that

$$H_n|0\rangle = 0 = \langle 0|H_{-n}, \quad n > 0. \quad (3-17)$$

For what follows, we also need to introduce the free fermion field operators

$$\psi(z) := \sum_k \psi_k z^k, \quad \psi^*(z) := \sum_k \psi_k^* z^{-k-1} dz. \quad (3-18)$$

For each $n \in \mathbb{N}^+$, define the “charge n ” vacuum vector labelled by the integer n :

$$\langle n| := \langle 0|\Psi_n^*, \quad |n\rangle := \Psi_n|0\rangle, \quad (3-19)$$

$$\begin{aligned} \Psi_n &= \psi_{n-1} \cdots \psi_1 \psi_0 \quad n > 0, & \Psi_n &= \psi_n^* \cdots \psi_{-2}^* \psi_{-1}^* \quad n < 0, \\ \Psi_n^* &= \psi_0^* \psi_1^* \cdots \psi_{n-1}^* \quad n > 0, & \Psi_n^* &= \psi_{-1} \psi_{-2} \cdots \psi_n \quad n < 0. \end{aligned} \quad (3-20)$$

Given any g satisfying the bilinear identity (3-14), the corresponding KP and two dimensional TL tau-functions [2] are defined as follows:

$$\tau_{KP}(n, \mathbf{t}) := \langle n|e^{H(\mathbf{t})}g|n\rangle, \quad (3-21)$$

$$\tau_{TL}(n, \mathbf{t}, \mathbf{t}^*) := \langle n|e^{H(\mathbf{t})}ge^{H^*(\mathbf{t}^*)}|n\rangle. \quad (3-22)$$

The first set of parameters appearing in $\tau_{KP}(n, \mathbf{t})$ are the usual KP flow parameters, while the parameters $\mathbf{t} = (t_1, t_2, \dots)$ and $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$ in $\tau_{TL}(n, \mathbf{t}, \mathbf{t}^*)$ are sometimes called the higher (two dimensional) Toda lattice times [2, 18]. (If we fix n and the parameters \mathbf{t}^* , then $\{\tau_{TL}(n, \mathbf{t}, \mathbf{t}^*)\}_{n \in \mathbb{N}}$ may also be viewed as KP tau-functions in the \mathbf{t} variables.) The first three parameters (t_1, t_2, t_3) are just the independent variables appearing in the KP equation [19]

$$4\partial_{t_1}\partial_{t_3}u = \partial_{t_1}^4 u + 3\partial_{t_2}^2 u + 3\partial_{t_1}^2 u^2 \quad (u = 2\partial_{t_1}^2 \log \tau). \quad (3-23)$$

4 KP tau-function $\tau_r(n, \mathbf{t}, \mathbf{t}^*)$

4.1 Schur function expansions.

For each choice of the parameters $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$ we define the operator

$$A(\mathbf{t}^*) := \sum_{m=1}^{\infty} t_m^* A_m, \quad (4-1)$$

where the A_k 's, which are defined as

$$A_k = \sum_{n=-\infty}^{\infty} \psi_{n-k}^* \psi_n r(n) r(n-1) \cdots r(n-k+1), \quad k = 1, 2, \dots, \quad (4-2)$$

commute amongst themselves

$$[A_k, A_l] = 0, \quad \forall j, k. \quad (4-3)$$

Using the explicit form of A_k and the anti-commutation relations (3-1) we obtain

$$e^{A(\mathbf{t}^*)} \psi(z) e^{-A(\mathbf{t}^*)} = e^{-\xi_r(\mathbf{t}^*, z^{-1})} \cdot \psi(z) \quad (4-4)$$

$$e^{A(\mathbf{t}^*)} \psi^*(z) e^{-A(\mathbf{t}^*)} = e^{\xi_{r'}(\mathbf{t}^*, z^{-1})} \cdot \psi^*(z), \quad (4-5)$$

where $\xi_r(\mathbf{t}^*, z^{-1})$ and $\xi_{r'}(\mathbf{t}^*, z^{-1})$ are operators on functions of the auxiliary parameter z , defined by

$$\xi_r(\mathbf{t}^*, z^{-1}) := \sum_{m=1}^{+\infty} t_m \left(\frac{1}{z} r(D) \right)^m, \quad \xi_{r'}(\mathbf{t}^*, z^{-1}) := \sum_{m=1}^{+\infty} t_m \left(\frac{1}{z} r'(D) \right)^m, \quad (4-6)$$

with

$$D := z \frac{d}{dz}, \quad r'(D) := r(-D). \quad (4-7)$$

The latter operator acts on functions of z according to the rule

$$r(D) \cdot z^n = r(n) z^n. \quad (4-8)$$

The exponents in (4-4), (4-5) are defined by their Taylor series.

Using relations (4-4), (4-5) and the fact that inside res_z the operator $\frac{1}{z} r(D)$ is the conjugate of $\frac{1}{z} r(-D) = \frac{1}{z} r'(D)$, we get

Lemma 4.1 *The fermionic operator $e^{A(\mathbf{t}^*)}$ satisfies the bilinear identity (3-14):*

$$\left[res_{z=0} \psi(z) \otimes \psi^*(z), e^{A(\mathbf{t}^*)} \otimes e^{A(\mathbf{t}^*)} \right] = 0. \quad (4-9)$$

This lemma follows from the general approach in [2].

By (4-9), the expression

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) := \langle n | e^{H(\mathbf{t})} e^{-A(\mathbf{t}^*)} | n \rangle \quad (4-10)$$

provides us, for each choice of n , r and \mathbf{t}^* , with a KP tau-function (3-21). For any given choice of r , the function $\tau_r(n, \mathbf{t}, \mathbf{t}^*)$, viewed as a function of all the variables, is a $2D$ Toda lattice tau-function.

Proposition 4.1

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) = \sum_{\lambda} r_{\lambda}(n) s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*), \quad (4-11)$$

where the operator valued exponential $e^{-A(\mathbf{t}^*)}$ inside the vacuum expectation value is defined by its Taylor series and the r.h.s is viewed as a formal series. (The sum in (4-11) should be understood as including the zero partition.)

Note that the tau-function (4-11) can be viewed as resulting from the action of additional symmetries [3, 15] on the vacuum tau-function. The variables \mathbf{t} play the role of KP flow parameters, and \mathbf{t}^* is a set of group parameters defining the exponential of a subalgebra of additional symmetries of KP (see [15]). Alternatively, (4-11) may be viewed as a tau-function of the two-dimensional Toda lattice [16, 18] with two sets of continuous variables \mathbf{t}, \mathbf{t}^* and one discrete variable n . Formula (4-11) is obviously symmetric with respect to $\mathbf{t} \leftrightarrow \mathbf{t}^*$.

4.2 The generalized Cauchy-Littlewood identity and a proof of Prop. 4.1

What is meant here by the generalized Cauchy-Littlewood identity is the following relation, which can be interpreted as a (double) generating function for the Schur functions.

$$e^{\sum_{m=1}^{\infty} m t_m t_m^*} = \sum_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) . \quad (4-12)$$

When the parameters $\mathbf{t} = (t_1, t_2, \dots)$ and $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$ are given by (1-8) in terms of two finite sets of variables (x_1, \dots, x_N) and (y_1, \dots, y_M) , this reduces to the usual form (1-7) of the Cauchy-Littlewood identity. This will be proved here using the fermionic Fock space and scalar product formulae discussed above. (In the previous papers [13, 14] only sketches of these proofs were presented.)

For the proof, we need a preliminary lemma [2].

Lemma 4.2 *For $-j_1 < \dots < -j_k < 0 \leq i_k < \dots < i_1$ the following formula is valid:*

$$\begin{aligned} \langle 0 | e^{H(\mathbf{t})} \psi_{-j_1}^* \dots \psi_{-j_k}^* \psi_{i_k} \dots \psi_{i_1} | 0 \rangle &= \langle n | e^{H(\mathbf{t})} \psi_{-j_1+n}^* \dots \psi_{-j_k+n}^* \psi_{i_k+n} \dots \psi_{i_1+n} | n \rangle \\ &= (-1)^{j_1 + \dots + j_k} s_{\lambda}(\mathbf{t}), \end{aligned} \quad (4-13)$$

where the partition $\lambda = (\lambda_1, \dots, \lambda_{j_1})$ is defined by

$$(\lambda_1, \dots, \lambda_{j_1}) = (i_1, \dots, i_k | j_1 - 1, \dots, j_k - 1). \quad (4-14)$$

(Here $(\dots | \dots)$ is the Frobenius notation for a partition.)

The proof of this lemma follows from a direct calculation, with the help of the Wick rule [2].

We now introduce the vectors (cf. [17])

$$|\lambda, n\rangle := \psi_{-j_1+n}^* \dots \psi_{-j_k+n}^* \psi_{i_k+n} \dots \psi_{i_1+n} | n \rangle \quad (4-15)$$

$$\langle \lambda, n | := \langle n | \psi_{i_1+n}^* \dots \psi_{i_k+n}^* \psi_{-j_k+n} \dots \psi_{-j_1+n} . \quad (4-16)$$

It follows that

$$\langle \lambda, n | \mu, m \rangle = \delta_{mn} \delta_{\lambda\mu} , \quad (4-17)$$

so the set of vectors (4-15) and (4-16) form orthonormal bases for the Fock space F and its dual F^* , respectively.

For any partition λ expressed in Frobenius form as (4-14), we define the integer

$$a(\lambda) := j_1 + \dots + j_k . \quad (4-18)$$

Then Lemma 4.2 and the orthogonality relations (4-17) imply the expansions

$$\langle n | e^{H(\mathbf{t})} = \sum_{\lambda} (-1)^{a(\lambda)} s_{\lambda}(\mathbf{t}) \langle \lambda, n | \quad (4-19)$$

and

$$e^{H^*(\mathbf{t}^*)} | n \rangle = \sum_{\lambda} |\lambda, n\rangle (-1)^{a(\lambda)} s_{\lambda}(\mathbf{t}^*) . \quad (4-20)$$

Proof of the generalized Cauchy-Littlewood identity.

Consider the vacuum TL tau function

$$\langle 0 | e^{H(\mathbf{t})} e^{H^*(\mathbf{t}^*)} | 0 \rangle . \quad (4-21)$$

By (4-19),(4-20) this is equal to the r.h.s. of (4-12):

$$\langle 0 | e^{H(\mathbf{t})} e^{H^*(\mathbf{t}^*)} | 0 \rangle = \sum_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) , \quad (4-22)$$

(where the sum again includes the zero partition).

It follows from the Heisenberg algebra commutation relation (3-16) that conjugation of e^{H-m} by e^{H_m} just translates the coefficient of the exponent by m . This, together with eqs. (3-17) implies that the vacuum TL tau function (4-21) is equal to the l.h.s. of (4-12):

$$\langle 0 | e^{H(\mathbf{t})} e^{H^*(\mathbf{t}^*)} | 0 \rangle = e^{\sum_{m=1}^{\infty} m t_m t_m^*} . \quad (4-23)$$

Proof of Prop. 4.1. Consider the Taylor series in all time variables t_1, t_2, \dots defined by

$$e^{A(\mathbf{t}^*)} | n \rangle = \sum_{n_1, n_2, \dots = 0}^{\infty} t_1^{n_1} t_2^{n_2} \dots A_1^{n_1} A_2^{n_2} \dots | n \rangle . \quad (4-24)$$

Since the A_k 's all commute, the order of factors on the r.h.s. is irrelevant. Each A_k has the structure

$$A_k = \sum_{m=-\infty}^{\infty} e_{m, m-k} , \quad (4-25)$$

where

$$e_{m, m-i} := \psi_m \psi_{m-i}^* r(m) r(m-1) \dots r(m-i+1) . \quad (4-26)$$

Recall that

$$\psi_m | n \rangle = 0, \quad m < n, \quad \psi_m^* | n \rangle = 0, \quad m \geq n . \quad (4-27)$$

For a given n , it is convenient to decompose $\widehat{gl}(\infty)$ (as a linear space, not as a Lie algebra) into the following direct sum

$$a = a^{++} \oplus a^{--} \oplus a^{+-} \oplus a^{-+}, \quad a \in \widehat{gl}(\infty) , \quad (4-28)$$

where

$$a^{++} = \sum_{i, k \geq n} : \psi_i \psi_k^* : a_{ik}, \quad a^{--} = \sum_{i, k < n} : \psi_i \psi_k^* : a_{ik}, \quad (4-29)$$

$$a^{+-} = \sum_{i \geq n, k < n} : \psi_i \psi_k^* : a_{ik}, \quad a^{-+} = \sum_{i < n, k \geq n} : \psi_i \psi_k^* : a_{ik} . \quad (4-30)$$

It follows that

$$[e_{i,j}^{+-}, e_{k,m}^{+-}] = 0, \quad [A_i^{+-}, A_k^{+-}] = 0 , \quad (4-31)$$

while

$$[e_{i,m}^{++}, e_{m,k}^{+-}] = e_{i,k}^{+-}, \quad [e_{i,m}^{+-}, e_{m,k}^{--}] = e_{i,k}^{+-} \quad (4-32)$$

Each term in the sum on the r.h.s. of (4-24) is a linear combination of monomials of the form

$$e_{i_1, k_1} \dots e_{i_N, k_N} | n \rangle \quad (4-33)$$

for some N . The product contains terms $e_{i,k}^{+-}, e_{i,k}^{++}, e_{i,k}^{--}$ (with various subindices i, k), where the last two annihilate the vacuum vector $|n\rangle$. Using the commutation relations (4-32) we put all elements $e_{i,k}^{++}, e_{i,k}^{--}$ to the right in (4-33), then reduce (4-33) to a product of only the $e_{i,k}^{+-}$ terms:

$$e_{i_1, k_1}^{+-} \cdots e_{i_M, k_M}^{+-} |n\rangle, \quad M \leq N. \quad (4-34)$$

Now consider (for $i > 0$)

$$H_{-i} = \sum_{m=-\infty}^{\infty} E_{m, m-i}, \quad (4-35)$$

where

$$E_{m, m-i} := \psi_m \psi_{m-i}^*, \quad (4-36)$$

in terms of which we may write

$$e_{m, m-i} := E_{m, m-i} r(m) r(m-1) \cdots r(m-i+1) \quad (4-37)$$

$$e_{i_1, k_1}^{+-} \cdots e_{i_M, k_M}^{+-} |n\rangle = r_\lambda(n) E_{i_1, k_1}^{+-} \cdots E_{i_M, k_M}^{+-} |n\rangle. \quad (4-38)$$

In the expression

$$E_{i_1, k_1}^{+-} \cdots E_{i_M, k_M}^{+-} |n\rangle, \quad (4-39)$$

note that $E_{i,k}^{+-} E_{j,m}^{+-} = E_{i,m}^{+-} E_{j,k}^{+-}$ and therefore one can apply the action of the permutation group and commutation relations (4-31) to reorder the first indices in decreasing order, i.e. $i_1 \geq i_2 \geq \cdots$ and the second indices such that $-k_1 \geq -k_2 \geq \cdots$. Then

$$e_{i_1, k_1}^{+-} \cdots e_{i_M, k_M}^{+-} |n\rangle = r_\lambda(n) E_{i_1, k_1}^{+-} \cdots E_{i_M, k_M}^{+-} |n\rangle = r_\lambda(n) |\lambda, n\rangle (-1)^M, \quad (4-40)$$

where the partition λ in Frobenius notation is

$$\lambda = (i_1, \dots, i_M | -k_1 - 1, \dots, -k_M - 1). \quad (4-41)$$

Since from (4-20) we have

$$e^{H^*(\mathbf{t}^*)} |n\rangle = \sum_{\lambda} |\lambda, n\rangle (-1)^{a(\lambda)} s_\lambda(\mathbf{t}^*), \quad (4-42)$$

we finally get

$$e^{A(\mathbf{t}^*)} |n\rangle = \sum_{\lambda} |\lambda, n\rangle (-1)^{a(\lambda)} r_\lambda(n) s_\lambda(\mathbf{t}^*), \quad (4-43)$$

which proves (4-11). From this we may also deduce the following result.

Proposition 4.2 [13]

$$\langle s_\lambda, s_\mu \rangle_{r,n} = \langle n | s_\lambda(\mathbf{H}) s_\mu(-\mathbf{A}) | n \rangle, \quad (4-44)$$

where the variables $(\gamma_1, \gamma_2, \dots)$ forming the arguments of the Schur functions (see (1-3)) are replaced by the operators

$$\mathbf{H} := \left(\frac{H_1}{1}, \frac{H_2}{2}, \frac{H_3}{3}, \dots \right), \quad \text{and} \quad -\mathbf{A} := \left(-\frac{A_1}{1}, -\frac{A_2}{2}, -\frac{A_3}{3}, \dots \right). \quad (4-45)$$

Proof. From the generalized Cauchy-Littlewood identity we have

$$\langle n | e^{H(\mathbf{t})} = \sum_{\lambda} s_{\lambda}(\mathbf{t}) \langle n | s_{\lambda}(\mathbf{H}), \quad e^{A(\mathbf{t}^*)} | n \rangle = \sum_{\mu} s_{\mu}(-\mathbf{A}) | n \rangle s_{\mu}(\mathbf{t}^*) \quad (4-46)$$

It follows from (4-43), (4-19) and (4-17) that formula (4-46) is equivalent to (4-47), given in the following proposition, which allows us to express $\tau_r(n, \mathbf{t}, \mathbf{t}^*)$ in terms of the scalar product $\langle \cdot, \cdot \rangle_{r,n}$.

Proposition 4.3 [13]

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) = \langle e^{\sum_{m=1}^{\infty} m t_m \gamma_m}, e^{\sum_{m=1}^{\infty} m t_m^* \gamma_m} \rangle_{r,n}, \quad (4-47)$$

where we consider the scalar product on the space of functions of variables γ , while \mathbf{t}, \mathbf{t}^* are viewed as parameters.

5 Matrix models

5.1 Normal matrix model.

We now consider an ensemble of $n \times n$ normal matrices M (i.e., matrices which commute with their Hermitian conjugates). The following integral defines the partition function for the normal matrix model.

$$\begin{aligned} I^{NM}(n, \mathbf{t}, \mathbf{t}^*; \mathbf{u}) &= \int dM dM^+ e^{\text{tr}(V_1(M) + V_2(M^+) - MM^+)} \\ &= C_n \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} |\Delta(\mathbf{z})|^2 e^{\sum_{i=1}^n (V_1(z_i) + V_2(\bar{z}_i) - z_i \bar{z}_i)} \prod_{i=1}^n dz_i d\bar{z}_i, \end{aligned} \quad (5-1)$$

where

$$dM = \prod_{i < k} d\Re M_{ik} d\Im M_{ik} \prod_{i=1}^n dM_{ii}. \quad (5-2)$$

(This model has applications, e.g., to the Laplacian growth problem [12].) The last integral in (5-1) is taken over n copies of the complex plane, (z_1, \dots, z_n) are the eigenvalues of the matrix M and C_n is a normalization factor coming from the matrix angular integration. Here V_1 and V_2 are the power series

$$V_1(z) := \sum_{m=1}^{\infty} t_m z^m, \quad V_2(z) := \sum_{m=1}^{\infty} t_m^* z^m. \quad (5-3)$$

Formula (2-10) then yields a series expansion of the integral (5-1) as:

$$I^{NM}(n, \mathbf{t}, \mathbf{t}^*; \mathbf{u}) = \langle e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m z_i^m}, e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m^* z_i^m} \rangle'_n = \sum_{\lambda} (n)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*). \quad (5-4)$$

This formula may be interpreted [7, 8] as a multi-dimensional analog of the Borel summation of the series on the RHS which, generally, is divergent.

5.2 Two-matrix model.

Let us evaluate the following partition function defined on an ensemble consisting of pairs (M_1, M_2) of $n \times n$ Hermitian matrices M_1 and $n \times n$ anti-Hermitian matrices M_2 .

$$I^{2MM}(n, \mathbf{t}, \mathbf{t}^*) = \int e^{\text{tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)} dM_1 dM_2 . \quad (5-5)$$

It is well-known [11, 9] that this integral reduces to the following one over the eigenvalues x_i and y_i of the matrices M_1 and M_2 respectively:

$$\tilde{C}_n \int_{\mathbb{R}} \int_{\mathbb{S}} \dots \int_{\mathbb{R}} \int_{\mathbb{S}} e^{\sum_{j=1}^n (\sum_{m=1}^{\infty} (t_m x_j^m + t_m^* y_j^m) - x_j y_j)} \Delta(\mathbf{x}) \Delta(\mathbf{y}) \prod_{j=1}^n dx_j dy_j . \quad (5-6)$$

with the normalization factor \tilde{C}_n proportional to the unitary group volume. It was shown in [6] that this integral is a two dimensional TL tau-function.

From (2-12), we obtain that the partition function for these random matrices is a series of hypergeometric type

$$I^{2MM}(n, \mathbf{t}, \mathbf{t}^*) = \frac{\tilde{C}_n}{(2\pi i)^n} < e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m z_i^m}, e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m^* z_i^m} >'_n = \frac{\tilde{C}_n}{(2\pi i)^n} \sum_{\lambda} (n)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) , \quad (5-7)$$

where we use the fact that, according to (2-1)

$$(n)_{\lambda} = \prod_{i,j \in \lambda} (n + j - i) = \frac{\Gamma(n+1+\lambda_1) \Gamma(n+\lambda_2) \dots \Gamma(\lambda_n)}{\Gamma(n+1) \Gamma(n) \dots \Gamma(1)} . \quad (5-8)$$

We thus have the following perturbation series

$$\frac{I^{2MM}(n, \mathbf{t}, \mathbf{t}^*)}{I^{2MM}(n, 0, 0)} = \sum_{\lambda} (n)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) . \quad (5-9)$$

When all higher parameters t_j, t_j^* with $j > 2$ vanish (i.e, the case of a Gaussian matrix integral) and $n = 1$, this series can easily be evaluated as (cf. [8])

$$I^{2MM}(1, t_1, t_2, 0, 0, \dots; t_1^*, t_2^*, 0, 0, \dots) = \sum_{m=0}^{\infty} m! h_m(\mathbf{t}) h_m(\mathbf{t}^*) = \frac{\exp \frac{t_1 t_1^* + t_2 (t_1^*)^2 + t_2^* (t_1)^2}{1 - 4t_2 t_2^*}}{\sqrt{1 - 4t_2 t_2^*}} . \quad (5-10)$$

5.3 Hermitian one-matrix model.

It was shown in [6] that the partition function for the Hermitian one-matrix model is a one dimensional Toda tau-function that satisfies the so-called Virasoro constraints. Here we consider the perturbation series for this model expressed in terms of Schur functions. Let M be a Hermitian $N \times N$ matrix.

The partition function for the Hermitian one-matrix model with an even quartic potential is

$$Z(N, g, g_4) = \int dM e^{-N \text{tr}(\frac{g}{2} M^2 + g_4 M^4)} . \quad (5-11)$$

We choose the normalization of the matrix integral in such a way that $Z(N, g, g_4)$ is equal to 1 when $g_4 = 0$. Let us evaluate the simplest perturbation terms for the one matrix model using the series (5-10) [8]. First, consider the partition function (5-5) for the two-matrix model and set all $t_k = 0$ except t_4 , and all $t_k^* = 0$ except t_2^* . Then (cf. [1, 4]) it is known that if we put

$$g_4 = -4N^{-1}t_4, \quad g = -(2Nt_2^*)^{-1}, \quad (5-12)$$

we obtain

$$I^{2MM}(0, 0, 0, t_4, 0, \dots; 0, t_2^*, 0, 0, \dots) = Z(N, g, g_4), \quad (5-13)$$

and therefore

$$\sum_{\lambda} (N)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) = \int dM e^{-N \text{tr}(\frac{g}{2} M^2 + g_4 M^4)} = Z(N, g, g_4). \quad (5-14)$$

Remark. It is well-known (see for instance [4]) that, according to the Feynman rules for the one matrix model: (a) to each propagator (double line) is associated a factor $1/(Ng)$ (which is $-2t_2^*$ in our notations) (b) to each four legged vertex is associated a factor $(-Ng_4)$ (which is $4t_4$ in our notations) (c) to each closed single line is associated a factor N . Therefore we may say that the factor $(N)_{\lambda}$ in (5-14) is responsible for closed lines, the factors $s_{\lambda}(\mathbf{t} = 0, t_2, 0, \dots)$ are responsible for propagators and the factors $s_{\lambda}(\mathbf{t}^*)$ are responsible for vertices. It follows that Feynman diagrams containing $k = |\lambda|/4$ vertices and $2k = |\lambda|/2$ propagators give the terms:

$$(Ng_4)^{|\lambda|/4} (Ng)^{-|\lambda|/2} \sum_{|\lambda|=4k} (N)_{\lambda} s_{\lambda}(0, 0, 0, 1, 0, \dots) s_{\lambda}(0, 1, 0, \dots), \quad (5-15)$$

where the number $|\lambda|$ is the weight of partition λ .

An explicit calculation of the first three nonvanishing terms (corresponding to partitions of weight 0, 4 and 8) yields the same result as the Feynman graph calculation (cf. [4])

$$Z(N, g, g_4) = 1 - \frac{g_4}{g^2} \left(\frac{N^2}{2} + \frac{1}{4} \right) + \frac{g_4^2}{g^4} (32N^4 + 320N^2 + 488) + O\left(\frac{g_4^3}{g^6}\right). \quad (5-16)$$

6 Expression for $\log \tau_r(n, \mathbf{t}, \mathbf{t}^*)$

Recall that a product of two Schur functions can be expanded as a sum [10]

$$s_{\lambda}(\mathbf{t}) s_{\mu}(\mathbf{t}) = \sum_{\nu} C_{\lambda\mu}^{\nu} s_{\nu}(\mathbf{t}), \quad (6-1)$$

where the coefficients $C_{\lambda\mu}^{\nu}$ may be calculated via the Littlewood-Richardson combinatorial rule [10], and the weight $|\nu|$ of the partitions in the sum is equal to the sum of the weight:

$$|\nu| = |\lambda| + |\mu|. \quad (6-2)$$

Writing (4-11) as $1 + \sum_{\lambda \neq 0} r_{\lambda}(n) s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*)$ and taking the log, we obtain

$$\log \tau_r(n, \mathbf{t}, \mathbf{t}^*) = \sum_{\lambda, \lambda' \neq 0} s_{\lambda}(\mathbf{t}) k_{\lambda\lambda'}(n) s_{\lambda'}(\mathbf{t}^*), \quad (6-3)$$

where $k_{\lambda\lambda'}$ is given by the following series

$$k_{\lambda\lambda'}(n) = \frac{1}{1} r_{\lambda}(n) \delta_{\lambda\lambda'} + \sum_{k \geq 1} \frac{1}{k} \sum_{\lambda_1, \dots, \lambda_k \neq 0} r_{\lambda_1}(n) \cdots r_{\lambda_k}(n) \sum_{\nu_1, \dots, \nu_{k-2}} C_{\nu_1, \dots, \nu_{k-2}}^{\lambda} \sum_{\nu'_1, \dots, \nu'_{k-2}} C_{\nu'_1, \dots, \nu'_{k-2}}^{\lambda'} , \quad (6-4)$$

with

$$\begin{aligned} C_{\nu_1, \dots, \nu_{k-2}}^{\lambda} &:= C_{\lambda_1 \lambda_2}^{\nu_1} C_{\nu_1 \lambda_3}^{\nu_2} \cdots C_{\nu_{k-3} \lambda_{k-1}}^{\nu_{k-2}} C_{\nu_{k-2} \lambda_k}^{\lambda} , \\ C_{\nu'_1, \dots, \nu'_{k-2}}^{\lambda'} &:= C_{\lambda_1 \lambda_2}^{\nu'_1} C_{\nu'_1 \lambda_3}^{\nu'_2} \cdots C_{\nu'_{k-3} \lambda_{k-1}}^{\nu'_{k-2}} C_{\nu'_{k-2} \lambda_k}^{\lambda'} . \end{aligned} \quad (6-5)$$

The first few terms are

$$\begin{aligned} k_{\lambda\lambda'}(n) = & \frac{1}{1} r_{\lambda}(n) \delta_{\lambda\lambda'} + \frac{1}{2} \sum_{\lambda_1, \lambda_2 \neq 0} r_{\lambda_1}(n) r_{\lambda_2}(n) C_{\lambda_1 \lambda_2}^{\lambda} C_{\lambda_1 \lambda_2}^{\lambda'} \\ & + \frac{1}{3} \sum_{\lambda_1, \lambda_2, \lambda_3 \neq 0} r_{\lambda_1}(n) r_{\lambda_2}(n) r_{\lambda_3}(n) \sum_{\nu} C_{\lambda_1 \lambda_2}^{\nu} C_{\nu \lambda_3}^{\lambda} \sum_{\nu'} C_{\lambda_1 \lambda_2}^{\nu'} C_{\nu' \lambda_3}^{\lambda'} + \cdots \end{aligned} \quad (6-6)$$

Note that, due to (6-2), each $k_{\lambda\lambda'}(n)$ consists of a sum over just a finite number of terms.

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